An Analysis of Counting

A THESIS

Presented in Partial Fulfillment of the Requirements for the Degree Master of Arts in the Graduate School of the University of Alberta

BY

HELEN FLETCHER KERR, B.A.,B.L.S.,B.Ed.
The University of Alberta

1964

PREFACE

An analysis of counting concerns itself with the natural numbers. Since these are the numbers of counting a possible starting point for such an investigation is a hypothetical account of the origin of natural numbers.

The purpose of such an hypothetical account, of course, will be of value only to the extent that it may shed some light upon the nature of the natural numbers themselves. There naturally will be no historical value in such an account. However, such a fictitious account may serve our purposes better than a strictly accurate account would that was based upon anthropological findings concerning the true origin of counting.

It is evident that the nature of numbers does not depend upon their origin in as much as even if we do discover their origin we have not necessarily thereby unveiled their nature. For example, some highly respected priest might have worked out our number-system in a world where there was no stability or permanency at all, and so great was his power that our system was adopted, and the number-words we use integrated into their language. In such a situation as this our number-system would be the one to be investigated just as it is now, and yet it would have had an entirely different origin from that which we believe ours to have had.

The plan of the thesis is as follows. In the first section, the writer attempts to show what facts of nature must be present and what primitive number sense man must have had in order for him to have conceived and invented the natural numbers.

Once these facts have been revealed and the counting system considered to be at hand, then it is examined itself to see what notations and operations must be present in it in order for it to be of value for the purpose or purposes it is intended to serve, and as well, what these purposes are.

The second main section of the paper deals with the number system as it stands. Here the object is to discover if the question so often asked and so seldom satisfactorily answered, namely, "What is number?" is in itself a meaningful one.

Finally the writer attempts to show what she considers to be the relationship, if any, between the fairly precise terms "one, two, three," etc. when applied to such concrete situations as, for example, pointing out cows and saying, "one cow, two cows, three cows," and so on, and the "one, two, three ..." which can be used quite independently of all observation and experiment.

The writer realizes that the section on recursive functions is very superficial and incomplete, and that the thesis must suffer to the extent that she is using this theory to some extent to back up her own position. However, since a thorough examination of this topic would entail knowledge of recursive number theory with which she has only a very slight knowledge, she feels justified in leaving a preliminary paper at this stage subject to further investigation in this area at some future time.

The writer acknowledges those sources used in the writing of this paper, so far as they've not been acknowledged in the text.

The greatest contributor of ideas and source of inspiration was Professor Richard Bosley of the Philosophy Department of the University of Alberta.

A number of ideas from R.L. Goodstein's Constructive Formalism, and F. Waismann's Introduction to Mathematical Thinking were used in the thesis. However, since the writer has worked these ideas into her own exposition, and has used them to back up her own theories, she is not at all sure whether or not these authors would acknowledge them as their own. To this extent, then, she must be responsible for them herself.

TABLE OF CONTENTS

Section Page

PREFACE.. 1

INTRODUCTION..................................... 4

A. A LOGICAL ACCOUNT OF COUNTING........... 7

I. Some improbable worlds............ 38

II Consistency....................... 48

III The concept of successor.......... 54

IV Notations, operations and where they led....................... 58

B. THE NATURE OF NUMBERS.................. 77

A SELECTED BIBLIOGRAPHY.................... 93

INTRODUCTION

Numbers are so much a part of our language that it is difficult to examine them objectively. We make use of numbers in every phase of our living. Without numbers we would not have the mathematics we now have, nor the sciences which use mathematics as their foundation. The foundation of mathematics itself seems, on the surface, very simple, namely, the natural numbers.

Counting is commonly acknowledged to give us the natural numbers from which all other numbers, real, rational, irrational and complex or imaginary, can be obtained. For example, here is a quotation from a standard text:

> ... we shall sketch a manner in which the real number system may be arrived at by definition from a postulate set for the much simpler and more basic system of natural numbers, the numbers of counting. In this way we shall make the consistency of the great bulk of mathematics depend upon that of the very fundamental system of natural numbers. The success of carrying through the program is credited to late nineteenth-century researches by Peano, Dedekind, and Cantor ...[1]

If man had never counted he likely never would have developed precise numbers beyond that of four. Investigations that have been carried out show that man, without special training, is unable to distinguish more than four objects in a group.

> Curr, who has made an extensive study of primitive Australia, holds that but few of the na-

[1] Eves, H. & Newsom, C.V. <u>An Introduction to the Foundations and Fundamental Concepts of Mathematics.</u> N.Y., Holt, Rinehart and Winston, 1961. p.195.

> tives are able to discern four, and that no Australian in his wild state can perceive seven.[1]

To get beyond four or five, man had to "count" objects, or at least tally them up in some way by matching the elements with a model set of objects, and thus arrive at, what we call, the "cardinal number" of the group.

This led to the distinction being drawn between cardinal and ordinal numbers. When one wishes to know how many objects or elements there are in a group, then one wants to know the cardinal number of the group. The cardinals give us the abstract numbers - Those numbers which are used without connection to any particular objects, as three, five, nine, etc.[2] and which have proven such an invaluable aid to mathematicians, scientists, and indeed to us all.

The ordinal numbers show the order of things in series. They are first, second, third, fourth, etc. Without the ordinal numbers we would not have the ordered series of numbers that we now have, and this ordering, as we shall see, is a most important part of our number system.

Beside numbers, we also have numerals which must be distinguished from numbers. Nagel and Newman point up the difference between numbers and numerals in this way:

---

[1] Dantzig, T. Number, The Language of Science. N.Y., Doubleday, 1954. p.5

[2] cf. McDowell, C.H. A Short Dictionary of Mathematics. N.Y., Phil. Library, 1957.

> A <u>numeral</u> is a sign, a linguistic expression, something which one can write down, erase, copy, and so on. A <u>number</u>, on the other hand, is something which a numeral <u>names</u> or <u>designates</u>, and which cannot literally be written down, erased, copied, and so on. Thus, we say that 10 is the <u>number</u> of our fingers, and, in making this statement, we are attributing a certain "property" to the class of our fingers; but it would evidently be absurd to say that this property is a numeral. Again, the number 10 is named by the Arabic numeral, '10', as well as by the Roman letter 'X'; these names are different, though they name the same number.[1]

This investigation is concerned with numbers - the natural numbers in particular. A good starting point for such an inquiry is an account of the origin of counting from a logical point of view.

[1] Nagel, E. & Newman, J.R. <u>Godel's Proof.</u> N.Y., Univ. Press, 1958. p.83.

A LOGICAL ACCOUNT OF COUNTING

Let us imaginatively develop the way counting might have arisen in a culture.

Suppose some tribe existed which had no number language. A native of this tribe, wanting to tell his neighbours at home how many cattle he has in his mountain pasture might first have tried picking up bits of ice from the mountains and matching these bits, one for one with his cows. However, when he arrives at his hut in the valley, he finds only wet pockets and still no way to display to his wife how many cattle he has.

The next time he tries chunks of mud. But this time, too, he is unsuccessful. When he arrives home this time he finds the mud has cracked and broken, and he doesn't know himself which is to count as one cow and which as two.

The third time, profiting from these experiences, he matches pebbles with cows, and this time he successfully demonstrates how many cows he has in his herd. Pebbles, he has found to be more durable than ice or mud, and thenceforth he and the other natives of the tribe use pebbles when they want to indicate how many things there are in any group or collection.

This native has arrived at the first step on the way to the use of numerals. That is, he has used pebbles to represent the members or elements of a group. The pebbles are suitable for the purpose, just as our numerals are

suitable for representing to us how many things there are in a group. However, the native could have made use of chalk marks on a stone, or notches in a stick, or tied knots in a rope to serve the same purpose, just as we could use numerals written on paper, or cast in iron, or marked on a stone, or written in a different notation altogether, such as Roman numerals. Just as our numerals are symbols to express numbers, so were their pebbles symbols to express how many members were in a particular group.

The pebbles are to these primitive people as our numerals are to us - not our numbers. They might change their pebbles for something else, or assign different values to each pebble, just as we might change our numerals or assign different values to each or any of them, and yet the numbers would not change.

Numerals are helpful to us in working out proofs in arithmetic, but it doesn't matter what these numerals are as long as they give us such proofs. E.g. $1 + 1 = 2$; $1 + 2 = 3$. The pebbles were useful to the natives in proving how many cattle they had in their herds, or sheep in their flocks, and it didn't matter what they used instead of pebbles as long as the substitute would give them such proofs.

Numbers, on the other hand, are quite different. We can change numerals at will, but not numbers.

Numbers are somehow essential to a group or collection in that if the number of a group changes then the group itself has changed. Other properties of things, such as colour, smell, or taste are not essential to things in this way.

The central character of this tale has used objects, namely pebbles, which are relatively durable, comparatively simple to carry about and which can be used in displaying how many things there are in less manageable groups. He is using pebbles as the model with which he can compare other groups. He has chosen the pebbles for the reasons mentioned above. Moreover, these pebbles not only can be used to show how many cattle he has in his herd, but they can be used to show how many objects there are in any other group - Although, of course, he may not realize this fact at this stage.

This man worked out this pebble-model system because he needed to solve the problem of how to keep track of how many cows he had in his herd. He wanted to know, for instance, if all the cattle (or the goats or sheep) were returned to the fold in the evening after a day in the pasture. Such groups have distinct members. Cows, goats, sheep and fish, etc. have some degree of stability and permanence, and he and the other tribesmen were able to identify and distinguish one of these members from another.

The pebbles serve only to show how many things there are in a group - They do no more than this. They don't give the other natives any information about the members of the group, such as whether they are animal, mineral or vegetable. They don't tell them how much these members weigh, what shape they are, how long, or how old. The pebbles only keep count. They don't describe the group in any other way, but they do describe it in this way. The pebbles only show how many distinct members one can distinguish in a group. As a matter of fact, we could imagine the native pointing out each goat, and for each such demonstration, dropping a pebble in his sack.

It is to be noted, though, that at this stage in the development of numerals, the pebbles cannot be said to be used in an abstract way. That is, each pebble does not stand for "one". Rather, each pebble stands for the cow, or the goat, or the sheep or whatever is being " mirrored" . The pebbles only picture the herd or the flock in a more permanent form than the individual's memory could do it for him. Thus, when he keeps track with his pebbles of his sheep that have gone to pasture then each pebble is thought of as a sheep. When he keeps track of the goats in the same way, each pebble is thought of as a goat. The man does not have to see what is common between a cow and a goat to use a

pebble in this way.

The pebbles can also be used in the economic life of the tribe. A native now can travel to a neighbouring tribe without having to take his cattle with him. Instead he can carry a bag of pebbles. When he arrives he can tell the chief of the tribe that he wishes to have this many pebbles-worth of cattle for his daughter's hand in marriage. The chief then will try to find one of his subjects who can match cows with the pebbles, and who also is willing to trade his cows in such a deal.

Again it is to be noted that there is nothing abstract about the pebbles at this stage. The only advantage in taking the pebbles over taking the cows themselves is that the father can make better time carrying pebbles than herding cows, and there's not as much danger of having some or all of them get lost.

So far there is no need for language to carry on these transactions. Simple addition could readily be carried out without language. For example, Native #1 could combine the two groups of pebbles and then show his friends how many cattle he has in his herd since he sold his daughter. He can do this, and the others can follow what he's doing as long as he and they know the rules, namely, matching pebbles with cows in a one-one correspondence. These are the rules that he's following in performing this procedure.

Other so-called operations can also be performed by these natives just using their pebbles in this matching manner.

For example, a native might want to know how many cows he has left if he trades this many pebbles-worth for a wife. In this case he will separate out a group of pebbles from the total pebbles in his cow-sack. He can also separate out a group of pebbles standing for the cows he has from the total number required in order to find out how many more cows he will need before he can buy the chief's daughter - Or, he can use this same procedure to find out, for example, how many more goats he has than his neighbour.

Some of the brighter natives might even develop a short-hand method of combining and separating pebbles. For instance, when they are combining a number of equal groups, they need only put down one group and then count those same pebbles as many times as there would have been equal groups.

Separating out equal groups, of course, would be more difficult. However, suppose a native has this many sheep: ////////// ////////// ////, and knows that this many: //////, will buy a lot of land. The procedure he might follow here would be to separate his pebbles in groups like this: ////// ////// ////// //////, and see how many groups he has, and hence how many lots of

land he can buy. If he ~~had~~ had this many sheep: ////// ////// ////// //, then he could explain that this means he has enough sheep to buy this many: ////, lots of land and he has still this many: //, sheep left over.

Indeed, at this point there appears to be no necessity of having a number language at all. At this point it seems that numerals might be the basis of arithmetic rather than numbers.

A monetary system could even be worked out without words. As these natives became more and more accustomed to dealing in pebbles rather than directly with what the pebbles represent, some value begins to be attached to the pebbles themselves.

At first, for instance, a cow might be thought to be worth this many, "/////" sheep. The tally marks or pebbles would only be thought of as mirroring the sheep. Indeed perhaps oblong pebbles might be used to tally up the sheep, round pebbles for cows, black pebbles for goats, etc. Gradually, however, as the pebbles are used more and more, the natives begin to use them, or bits of some metal which they value, in exchange for the cow. They know that they can use these bits of metal in exchange again for something they want.

Finally the cow is thought of as this many bits of precious metal:"/////", with no thought given to the sheep involved. We can now say that abstraction has

entered the picture. Now the pebbles or metal pieces are thought of as things that can be applied generally, and are not tied down to what they represent. Now the same pebbles can be used to keep track of anything - Cows, goats, sheep, or even other pebbles.

As soon as these natives have such objects to serve as a model some become interested in arranging the pebbles in various combinations. For example, one of the wise young men of the tribe might try matching some of his pebbles from his sack with trees, or cows, or even pebbles in another group. The pebbles he had used are the pebbles in the "game" . The ones still in his bag are "dead" - They cannot be allowed to enter the "game". He finds that when he matches the pebbles in the game with all the elements in all the groups he has used, that invariably he must use all and only all the pebbles that he previously used when he matched pebbles with the elements in the individual groups.

Our tribesman distinguishes between the pebbles he had used in the game, or brought to it, and the ones that have not been brought to the game. He can explain what each means. For example, he can say, "This pebble is to stand for the white goat with the black spots which is always escaping from the goat-herder." When he says this, all who hear him understand him. The goat-herder knows that his master will be careful to see

that the goat in question is returned home. In the same way the native could refer to each of the "pieces" in the "game" in his language. For example, "I sold this many animals to-day" ; "This is how much the sheep cost ..." and so on. In each case the meaning would be clear to all those who understood the language.

That is, even though the pebbles can be used generally, they must be able to be applied to these natives' everyday world. It is this application which gives them meaning, and this meaning will be clear to all who understand the pebble-counting or matching procedure and how the pebbles may be referred to in the language.

However, this man knows that if he allows any of the "dead" pieces to enter the game, then he will upset all the pieces which are "in gear", so to speak, - That is, the pieces which are taking a part in the game.

The pebbles in the "game" are playing a certain role which the "dead" ones do not play. However, once these other pebbles are legitimately brought into the game, by being required to represent new elements of a group, or rising costs, and so on, then they too play a role in the game.

This wise native knows that each of the pebbles, with which he is playing, not only stands for a cow, a tree, etc., but that it is part of his whole system of "arranging pebbles" in his matching game.

Incidentally these natives are also learning how to operate, or perform operations with these pebbles as we have seen above. For instance, to stick to addition here, they are learning that a count of objects in two or more separate groups requires the same number of pebbles as a through count of all the objects in all the groups. Moreover, it doesn't matter which objects from which group they count first in the through count, the total will be the same. It doesn't matter either, with which object they begin or end the count - Again the through count will be the same.

The mathematically-minded native, along with a few others of like disposition become so adept at the game of pebbles that they begin to use more short-cuts in their "computations". Instead of using pebbles for all matching, they use their fingers and when they have used all fingers on both hands, then they allow a large pebble to "enter the game". They proceed in this fashion until they have used up, for example, this many large pebbles: "@@@", and have still this many cows left over: "/////". They then match small pebbles with left-over cows and put these with the large pebbles.

In the case where there are "@@@@@" large pebbles, for instance, where "@" represents as many fingers as one has on both hands, and there are none left over, then no small pebbles are put in the sack.

These wise men of the tribe are giving a base to their "number-system", viz. the base 10. The reason for this base rather than 13, 15, or any other number, is due to the fact that they have ten fingers which are readily available for counting purposes.

It is interesting to note that Aristotle comments on this in his Book of Problems. He asks, "Why do all peoples, barbarians as well as Greeks, count by tens and not otherwide?" To this question he replies that this depends on the fact that the hands of man have in all ten fingers.[1]

The wise men of the tribe are beginning to lay the foundations of abstraction as well.

At this point it might not be out of place to explain what is meant by abstraction.

Runes' Dictionary of Philosophy gives this definition of abstraction:

> The process of ideally separating a partial aspect or quality from a total object. Also the result or product of mental abstraction. Abstraction, which concentrates its attention on a single aspect, differs from analysis which considers all aspects on a par.

Now let us see how these natives are using abstraction. An example is given when they pair off pebbles with a group of men, and put these pebbles in a bag, and then match pebbles with a group of horses, and put these pebbles in the same bag. The wise men know

---

[1] Aristotle. Book of Problems. (As quoted in Logsdon, M.L. A Mathematician Explains. Chic.,Univ.of Chic.,1960. p.15)

that all that these pebbles will show is how many animals there are, and not how many centaurs!

The partial aspect or quality which was separated by these natives was animality. The horses and men were both animals in contrast to the horses and the trees, for instance.

Here it can be seen that the first principles of abstraction are being learned. Just because one is concerned with some common characteristics and is displaying how many "animals" there are coming down the mountain track, that therefore it does not follow that an unique animal exists described by these common characteristics. That is, the use one makes of the abstract general term "animal" does not mean that "animal" describes the object in question, although some of the lesswise natives may believe that it does.

Of course, these scribes revel in the power and prestige their knowledge and skill gives them among the other natives, and so they don't divulge their secrets to them. They allow those who wish to do so to believe that the use of the pebbles and their manipulation are done in accordance with the gods' laws, and according to the order which exists in heaven and which the gods themselves have divulged to this select few.

The gullible ones in the tribe think that the future can be foretold and controlled by means of the pebbles,

For example, the scribe who goes around "counting" cattle, tells a man that he must add this many, "/////", pebbles to his cow-sack, since his herd has increased by that many since the last census. The owner of the cows assumes that he can control and foretell how many cattle he will have in his herd by means of the pebbles alone. He merely keeps adding groups of this many, "/////", pebbles to his sack, and sneaks in an extra one or two which he believes will result in that many more cattle being added to his herd. He then displays his pebbles to his friends to show them how many cattle he will have.

The scribe, of course, takes quite a different view of this operation. He realizes the use he, himself, has been making of the pebbles and does not attempt to read beyond the pebbles in the "game" or which can be legitimately brought into it. Moreover, he doesn't think the native is interfering with the god's order when he illegally allows some "dead" pebbles to enter the sack - He thinks the native is cheating!

So we find that some of the natives, particularly those not "in the know", confuse the pebbles with the use that is being made of them, as in the second example, and they also confuse the use of the pebbles with the way they describe the object, as in the first example. We can find examples of such confusion today.

Examples of the second type of confusion can be found in the misuse of statistics. Mark Twain gives us a humorous example of this confusion in his Life on the Mississippi.

> In the space of 176 years the Lower Mississippi has shortened itself 242 miles. This is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the old Silurian Period, just a million years ago next November, the River was upward of 1,200,000 miles long, and stuck out over the Gulf of Mexico like a fishing rod. And by the same token, any person can see that 742 years from now the Lower Mississippi will be only a mile and three-quarter long, and Cairo and New Orleans will have joined their streets together ...

Examples of the first type of confusion can be found in philosophy, among other places. For example, since we, at the present time at least, make use of and refer to "classes", an abstract term, in mathematics and other sciences, some philosophers and mathematicians say that classes exist. Platonists take the view that mathematicians "discover" the "objects" of mathematics, they don't invent them. If this is the case then, these objects must "exist" before they are "discovered". Kurt Gödel, in his preface to The Philosophy of Bertrand Russell (p.137) appears to hold such a view when he says:

> Classes and concepts may ... be conceived as real objects ... existing independently of our definitions and constructions. It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence.

The scribes, or high-priests, as they now have become, realizing their limitations in identifying how many things there are in a collection by observation alone, use the pebbles and the pairing-off process as a criterion of correctness when discrepancies occur in answering the question, "How many?" If two scribes come up with a different number of pebbles when keeping track of cows, for instance, then the priests try to find the cause of the discrepancy. Perhaps two cows were paired off with one pebble in one of the "counts" or perhaps one cow wandered away after the first native made his collection of stones. They check by having members of the esoteric circle repeat the pairing-off process with pebbles, and they accept the answer of the majority.

The natives, by agreement, are using their "model" set of objects (the pebbles), and the pairing-off process as a criterion of correctness. They have found that they can't estimate how many cows there are in a herd of cattle, nor fish in a catch, and so on, by sight alone. They have chosen the pebbles to help them keep track because they are assured of the relative durability of the pebbles, and their persistence through space and time, and, of course, they find them fairly easy to use in such transactions. They find that they can be used for any number of transactions which they may encounter. For these reasons, the pebbles are accepted, whereas the bits of

ice and the chunks of mud were rejected. The ice had melted and so did not persist in the same form in space and time, and the mud broke into pieces and so was not durable enough for the use to which the natives wished to put it.

Eventually one of the brighter lads of the tribe complains about the awkwardness and the inconvenience of carrying a supply of pebbles around to perform this operation, and he invents a shorthand method of accomplishing the same result. He makes use of language instead of pebbles. He <u>says</u> how many things there are instead of just displaying how many.

Now it is quite possible that the tribe's language had in it the words for "ug", "oo-pah" and "me-tah", and possibly "velly", which might correspond to our "one, "two" , " three", and "many". Anthropologists point out that all primitive people had some number-words in their language, although the most primitive did not go beyond three, and many.

Conant has this to say on this matter:

> Among the barbarous tribes whose languages have been studied, even in a most cursory manner, none have ever been discovered which did not show some familiarity with the number concept. The knowledge thus indicated has often proved to be most limited; not extending beyond the numbers 1 and 2, or 1, 2, and 3.... But even here, rudimentary as the number sense undoubtedly is, it is not wholly lacking; and some indirect expression, or some form of circumolocution, shows a conception of the differences between one and two, or at least, between <u>one</u> and <u>many</u>.[1]

[1]Conant, L.L. "Counting". (In Newman, J.R. <u>The World of Mathematics</u>. N.Y., Simon & Schuster, 1956. v.1, p.432.

And again:

> We know of no language in which the suggestion of number does not appear, and we must admit that the words which give expression to the number sense would be among the early words to be formed in any language. They express ideas which are, at first, wholly concrete, which are of the greatest possible simplicity, and which seem in many ways to be clearly understood, even by the higher orders of the brute creation. The origin of number would in itself, then, appear to lie beyond the proper limits of inquiry; and the primitive conception of number to be fundamental with human thought.[1]

The writer believes that the question "How many?" was perfectly well understood long before "numbers", as we know them, were invented. The answer to the question, however, was not the one that we customarily give, although we may do so at some times. For instance, when asked how many dollars one has on hand, he might very well reply, "Oh, just enough to keep eating this week." So the native might reply to the same question before the advent of number, "Oh, just enough goats to supply milk for me and my family".

The purpose of this investigation is not to study the possibility of saying how many or how much by saying "Just enough to feed myself and no more", but of giving the answer, "The number of goats I have is equal to ten", or merely "Ten" to the question, "How many goats have you?

Apparently then, this tribe of natives had the question "How many?", and they could supply an answer to the

---

[1] _ibid_. p.433

question as well. For instance, when asked how many goats are in his herd, a man might reply "Enough to support another wife", while another might respond to the same question with "Not enough for me to hold a potlatch on!"

These answers served the questioner and the questioned very well until trade became important to themselves and to the tribe. Then more precise answers were required.

Actually it was man's need to state precisely how many things there were in groups such as herds of cattle, in which there were distinct members, which eventually gave rise, via fingers and pebbles to number-words.

However, before they got to the number-words themselves, the natives invented this pebble-system which was convenient to use, and served sort of as a mirror image of the herd of cows, flocks of sheep and goats, or what have you?

It is important to note that these groups must have distinct members - Members that can be distinguished one from another, so that they can be matched with the pebbles or with tally marks in the sand or notched on a tree.

Let us consider the natives putting together two or more groups of apples, When asked, "Now how many have you?" the native can respond by showing this many: "//////////". This response wouldn't make sense if he

were putting together heaps of mashed potatoes, nor would it make sense if he were pouring water into a vessel - In the latter cases there's not a one-one correspondence for the heaps of mashed potatoes - It's just one big heap - Nor is there a one to one correspondence for the partitions of water in the vessel - It's just a volume of water.

"//////////" is somewhat like a snapshot of the cows where each cow can be distinguished in the snap. Just as a check can be made by comparing the picture with the cows one by one to see if the snap is correct - That is, that the snap shows all and only all the cows, so the tally marks can be checked with the apples in the group with which they have been matched, to see if the tally marks are correct. Once this procedure is accepted and acknowledged as the criterion of correctness by the natives, then these marks or pebbles can be used to see whether or not the group itself remains stable.

This is as far as the natives have got with their "number-system". The natives can now check to see whether or not they were right with their pebbles and their cows by having other tribesmen repeat the process of matching pebbles with the objects of the group making sure that they follow the rule of matching one and only one pebble with one and only one cow.

The next native genius in the tribe goes beyond the

few number words and these inconclusive answers - E.g., "Just enough to buy another wife," to the question "How many?" He adapts his use of pebbles to language, and makes use of words in language rather than using pebbles in his counting system. He says that from now on he will say "ug" wherever he had previously put down a single pebble when he was keeping track of things. "Oo-pah" will be said wherever he had previously put down _ug_ and _ug_ pebbles. "me-tah", will be said instead of using _ug_ and _ug_ and _ug_ pebbles, and then follows "um-lah", "seequa", "seen-ta", "sarta", "hapta", "nona" and "deca" , as _ug_ pebble is added to the group each time the next word is said in that order. He stops at _deca_ because he has sometimes used his fingers when keeping track of things, and as he has found fingers a fairly good substitute, as discussed above, he models his new vocabulary after them. He makes _deca_ the base for his number-system.

What this investigation must do is try to discover what makes these words more precise answers to the question "How many?" in contrast to the inconclusiveness, if such it is, of the answer "Not enough to eat on", and what advantages, if any, the words have over the pebbles. We have found that the pebbles were more convenient objects to handle in trading transactions than the actual objects which were to be traded. But we also found another advantage in dealing with pebbles rather than the

actual persons, things or animals being dealt with, in that the pebbles could be used generally. They could be used for any objects whatsoever, and need not be restricted to members of a particular group. Thus, the answer to the question, "How many sheep have you?" might have been, "Not enough to hold a potlatch on". This answer would only serve to answer this question, and perhaps a few more of the same nature. For example, such an answer would not serve its user well when asked how many moons he had travelled to visit the neighbouring tribe, or spent in hunting buffalo. Pebbles, of course, could be used to display how many moons were spent on such occasions. Moreover, the answer, "Not enough to hold a potlatch on," might have a different meaning in one tribe than in another. It is not a standardized answer. But once the procedure of matching pebbles in the manner described above is commonly accepted, then the pebbles can be used for standardized answers.

What must be noted however, is that it would be a false analogy now to compare the pebbles or tally marks with snapshots of reality. They are now more like standardized measures (e.g. a yardstick or a meter stick) which can be used to measure the dimensions of any stable object.

This very generality and standardization made it practical for the business man to carry the pebbles

about to use in commerce rather than dealing with the objects in question themselves.

Perhaps these number words are not any more precise than the seemingly inconclusive answers given, rather it may be their convenience and their generality which wins the natives over to using them.

The number-words gradually took the place of the pebbles, but served man in the same way as the pebbles, in that they could be applied generally. Just as abstract uses were possible with the pebbles, so also with number-words.

These natives have abstracted pure number-words from what might once have been taken as the names for these and not for those things. For example, compare our use of "Jim" and "he". "Jim" refers to this individual and not to that one. "He", on the other hand, refers to either or both, or any number of others. We can make use of "he" in a wide variety of contexts just so long as we know the rules of the language that allow for the use of the third person singular masculine pronoun, and also the rules that allow for substitutions in place of "he" in sentences. Just as we previously saw that the pebbles used to be used for this flock of sheep and not that herd of goats, or even that flock of sheep, they gradually came to be used for each flock or herd or all of them. Once the rules were made clear for the

use of pebbles and what things could be substituted for them, then the pebbles could be used in this general way.

The number-words evolved in the same way. People in this tribe now use words - the number-words, <u>ug</u>, etc., as they had formerly used pebbles when they let them stand for anything whatsoever.

As the natives become more familiar with using pebbles for keeping track of things, gradually the pebble was not thought of as a sheep, or a goat, as we have pointed out above, but as <u>ug</u>. The people had noticed that that animal was <u>ug</u> animal, the animal next to it was also <u>ug</u> animal, and so on. To keep track of the <u>ug</u> and <u>ug</u> and <u>ug</u> animals the natives used fingers or pebbles.

Again it is to be noted that the use they made of the pebbles can be compared with the use we make of numerals. The pebbles, e.g. "@@@///", just like our numerals, "33" enabled them to arrive at the <u>cardinal number</u> of the collection.

The inventor of this new system is very pleased with himself, and yet he realizes that his new "invention" will be of little value to him unless he can get others to use it as well. As a private language it has no value - Why, he would still have to use pebbles in his trading transactions, and in order to convey information to others about how many things there are in groups.

The beauty of this new system, if it were widely known and accepted, would be its facility in conveying such information as this. It could be used wherever the pebbles could be used, and so could be used in a wide variety of transactions, all of which the natives find necessary in their social and economic life. It could easily be taught to other tribes, and so would eliminate the sacks of pebbles the negotiators must now carry about whenever they make journeys for trading purposes. Moreover, these <u>words</u> are even more durable than the pebbles. Why, it was found that pebbles would wear away, or occasionally break in two in the midst of a transaction. Pebbles could be stolen, and switched from one group to another by canny tribesmen - But words! - Why, if they were accepted and became common property, they could be passed down from generation to generation with the rest of the language. As long as members of the tribes who used these words lived, so would the words.

The high priests fear that once these words take on the role in language that the pebbles had in the other keeping-track-of-things system, then the natives would no longer worship the pebbles, and the priests themselves might be in danger of losing some of their prestige. However, the scribe assures them that the natives will learn to respect the number-words in the same way, and attribute magical powers to them just as they had done with

the pebbles. As a matter of fact, they may be even able to dupe more of the natives since the priests would be able to do complex calculations without using any material objects at all.

In Quine's terminology, these natives have made the "semantic ascent".[1] They have made the transition from <u>showing how many</u> to <u>saying how many</u>. But the words now are part of this language - They are part of the whole system, and they must be used in accordance with the rules of grammar which admits them to the "game". It is the whole system which must be taken into consideration in seeing the use of the number-words, just as it was with the pebble-tracking system and the pebbles in the game and the "dead" ones - It is not just the signification of the words by themselves. As a matter of fact, we have found one reason why the number-words are important, since they serve to tie the numerals up with man's other knowledge.

Paralleling the development of the use of pebbles, at first the number-words were always used to refer to concrete objects. However, as the words become more familiar, and particularly after the invention of money, the natives got accustomed to thinking of the object in terms of what was substituted for it. For example, a goat might be thought of in terms of <u>deca-deca-deca ochas</u> - its purchasing price. From this it was an easy step

[1] Quine, W.V.O. <u>Word and Object</u>. Cambridge, Mass., M.I.T., c1960. p.271-276.

to drop the object which went with the number or numeral, and the number is thought of alone. The transition then from concrete numbers to abstract numbers had taken place.

But to come back again to our tribe of natives who are making such progress in developing mathematics. It is chiefly its utility and ease of use that delights the natives when they are instructed in the use of these number-words. Now when they want to tell their friends how many cows they have, there is no need to dig out the right number of pebbles from their pebble-bags and display all these pebbles, they merely have to say, for instance, "There are <u>deca</u>-<u>deca</u>-<u>seenta</u> cows". These words convey to their listeners that the man has as many cows as he has fingers on his two hands, toes on his two feet, fingers on one hand more, and noses on his face.

The inventor stresses the importance of saying the right word for the right occasion, and gradually through imitation the others do the same. Thus, one must always say "ug" when previously one had used his little finger on one hand for keeping track of an object, or where previously he had used <u>ug</u> pebble. "Oo-pah" must be used for <u>ug</u> and <u>ug</u> more, and so on. Moreover, the genius even points out that these words can be put in particular order - It can be used for counting them as well as showing how many objects there are in a group. He demonstrates and justifies this order by appealing to a device that

all the natives have used, namely, turning down fingers in a certain order when "counting" things.[1] So now a native can count using words instead of fingers. He can say "ug" where previously he had turned down the little finger on his left hand; he can say "oo-pah" where before he had turned the "ring" finger on the left hand down; he can say "me-tah" were previously he had turned down the middle finger, and so on. These words will now have a certain order based on each one meaning ug more than the word just before it. Ordinal numbers as well as cardinal numbers have been invented.

Some of the natives find this order difficult to grasp, accustomed as they are to dealing with pebbles. It didn't matter what pebbles they had dug out of their bags as long as they matched one to one with the cows. These stupider individuals argue that the same principle should apply to the words. Why not just say the word for pebble as many times as one would put down pebbles? Supposing this word were "roca", then one could say there

[1] "In the method of finger counting employed by savages a considerable degree of uniformity has been observed. Not only does he use his fingers to assist him in his tally, but he almost always begins with the little finger of his left hand, thence proceeding towards the thumb, which is 5.... but oftener the fingers of the right hand are used, with a reversal of the order previously employed; i.e., the thumb denotes 6, the index finger 7, and on to the little finger, which completes the count to 10." - Conant, L.E., op.cit. p.437.

are roca, roca, roca, roca, roca, roca, roca corws in that tree.

An "Alice" in the tribe soon shows the difficulty in this method when she retorts: "I don't know how many crows there are - I lost count!"

> "What's one and one and one and one and one and one and one and one?"
> "I don't know," said Alice, "I lost count."
> "She can't do addition," said the Red Queen.
> - Alice in Wonderland.

Not only do the audibly distinct words have the advantage of enabling "Alice" and the others to keep count, but they also save them the trouble of having to say the same word so many times. As we have pointed out above, these number-words automatically keep track for us. The number-words of this tribe enable the natives to keep track automatically too. For example, "me-tah" tells them that they have counted ug and oo-pah and have gone on ug more. "Um-lah" tells them that they have counted ug, oo-pah, me-tah, and gone on ug more. The word " um-lah" itself, for instance, indicates to those who understand it that there are this many, ////, things being counted, just as surely as the four tally marks do, or saying the four words, "ug, oo-pah, me-tah, and um-lah" do.

In addition to this, "um-lah" is distinct. If it is ug and ug and ug and ug it can be nothing else, and the next one following it among the numbers whether it

is called "see-qua" or simply "the next number after umlah" can be nothing but <u>ug</u> and <u>ug</u> and <u>ug</u> and <u>ug</u> and <u>ug</u>, or this many tally marks: /////.

Actually this is a very sophisticated concept and much instruction has to be given by the high-priests and scribes in order to get it across to at least some of the natives.

This native genius has given to his tribe and its members for generations to come, words that are audibly distinct, that tell their users how many elements there are in any collection (which, of course, will be relatively small, as there is no need for large numbers at this stage in the tribe's development), and which can be used to refer to any group with distinct elements whatsoever.

Why the tribe has accepted this language rather than torturing the inventor is that its wise men see how useful and convenient it is to them in their simple trading transactions and in their social life.

Certainly this genius has used abstraction in that he has invented words that can be used to keep track of any kind of objects that can be distinguished one from the other. But this is not new to the words alone. He used the same sort of abstraction with the pebbles. Why did he abstract the cows, the trees, the fish, the girls, the goats, etc., and just leave the members in the collec-

tions? One reason, of course, is that he had the senses which enabled him to distinguish objects one from another and to say which was the same and which was different. A possible reason why he made words, or used pebbles, to show how many of these distinct elements there were in a collection is that his environment made this information important to him.

Many writers have emphasized the importance to man of being able to abstract the unnecessary for the problem in hand. If he were unable to forget all the irrelevant details, his mind would be so lost in these details that he would never go on to grasp the significance of the whole. We also hear from time to time that only the intellectually capable are able to do abstract reasoning. This may not be true. It is quite likely that every one of us, the dull as well as the bright, indulges in abstracting details.

Morris Kline would agree with this. We find him saying in his Mathematics and the Physical World (p.26):

> The process of abstraction is far more natural and realistic than appears at first sight. Every young boy envisages the perfect girl, and girls entertain a similar thought about boys. Do these perfect beings exist in the physical world? Almost certainly not. But aside from the pleasures that contemplation of these ideals afford, the thoughts themselves may find application to problems and actions in the world of experience.

Our native genius has worked out this language be-

cause he first of all realized his limitations in not being able to discern more than three elements in a collection, and secondly, he did have the concept of numbers up to three, and as well the concept of <u>many</u>. He found it necessary to be more precise in answering the question, "How many?" than merely by replying "Oh, just enough to live on", or something of the sort. He found it necessary to give more precise answers than this sort when keeping track of the number of cows in his herd, in sharing pieces of meat and fish with other members of the tribe, in knowing how many fish he must catch before he can supply each member of his family with enough food for a certain number of days. Other natives find such a system useful as it enables them to keep track of days and seasons so they'll know when to plant the corn, and so on, or merely to be able to tell when a religious festival is coming up. The chief of the tribe finds this system useful when he collects the "taxes" from his subjects, and also when he is mustering an army or wants to know the number of his enemy's warriors.

The natives have abstracted what they found necessary to abstract in order to have a system which they can make use of in a number of their daily activities. The number system they use has evolved from experience, and has been adopted because it is found useful in satisfying their needs.

I. SOME IMPROBABLE WORLDS

Suppose that their world had been an entirely different one from the one in which they lived. Imagine, for example, that it was one in which nothing lasted longer than the time it takes to count to ten. That is, before one reached eleven the herd of cows would have disappeared. In such a world there would be no need of numbers or numerals beyond 10.

Now it is conceivable, as we have previously surmised, that a system of numbers such as ours might have been invented by one of the high-priests of the tribe, and that his power was so great that the system and its words were adopted into the language of the tribe. Of course, adaptations would have to be made so such words as "fifty", "hundred", etc. could be used in the language. Since there would be no synonyms for such words in their language it is quite likely that things around these people would have to be looked at in a different way than the way in which they had been previously regarded. For example, perhaps each stage of the cow would be counted while it was present. Each cow might even be considered to have 10 stages for its duration, or something like this might be evoked to use up some of the slack in numbers. Philosophers could investigate the nature of the sacred system, and the Platonists might even come out on top with their explanations.

The practical man, however, who needs the mathema-

tics in his everyday affairs and other investigations, might pay homage to the sacred system but would likely inwardly rebel against it. Gradually the dissenters would invent and use a number-system for practical purposes that did not go beyohd 10. Over the years the new system's number-words and meanings would take priority over the old words in the language. Since the new system's would be applicable to their world then its words would have meaning for its users. Explanations could be given for the number-words and the operations in terms of objects about these people. Since the older system is not applicable its words would gradually become obsolete.

Again, let us imagine an expanding infinite universe in which all objects (stars, and their satellites) meteors, comets, planetoids, and what not, are not only hurtling after each other, but are also hurtling headlong through space at some fantastic number of light years per second (in our reckoning of time, of course).

One of these objects hurtling through space is a twisted strip[1] composed of a finite number of conscious specks, joined contiguously like indistinguishable beads on a single thread. An added feature of this strip is the rate of attraction that these particles have for one another. All are attracted to one another with such force that they remain in their respective positions

[1]The Möbius strip.

until another particle from space lands on the strip with such force that it pushes one of the specks already there out and takes its place, and then and only then does it become conscious itself.

Each of the particles forming part of this strip is assumed to have its own conscious feelings, and yet is unable to say just where it ends and its neighbours begin. Nor can any one of them say which is the front and which is the end, which is the inside and which is the outside, which is the right or which is the left of the strip.

In this strip of conscious particles there is nothing to count as we know things that can be counted, at least as far as these beings are concerned. That is to say, there are no objects which these specks can distinguish one from another. There are no distinct pulse beats, or anything of this nature by which a discrete number series could originate. There are no nights, no seasons.

Moreover, on such a strip there would apparently be no motion as such, if by motion is meant the change in position of a physical object from one point to another. What appears as motion to us in their world (particles being knocked out by other particles) would appear as change in time to them (coming to life and dying), were they able to distinguish this change in time with their senses.

These specks are able to beam out signals like radio stations, but at first there are no distinct "beeps" - Just a continuous "eeeeeee..." like a short-wave station continually on the air.

The question that comes to mind is, What if any arithmetic will develop on such a strip? To answer this question we would likely ask another; What application is there for arithmetic on such a strip?

Suppose that one of the conscious particles noticing its own consciousness wonders if it composes the whole strip, or if there are more conscious particles on the strip, and if so, what part of the strip each constitutes.

Now it is to be noted that there is a finite number of these particles composing the strip. This number is already there even though the "star" has no way of knowing this number, nor of putting pebbles or anything else in one-one correspondence with it. In order for this one particle to know how many more <u>ones</u> there are, it has to get some way of transforming the number that is already there into a notation that it can understand.

We'll suppose that there is a way for each particle to turn off its signal, although it has never happened that all have been turned off at the same time. The "star" understands that it can only discover how many particles there are if it can distinguish one source of

communication from another. All messages must be stopped and then only a single source must be heard from at a time.

That is, this "star" particle is putting forth the notion of one and the next one and the next one - Without which notion no arithmetic at all could develop.

It accomplishes this silencing task after many unsuccessful tries, and finally through example and repetition it manages to get each particle to flash a "beep" in turn. The next step is to flash "*", itself, the particle immediately next it flashes "**", and the next particle "***" and so on. That is, it is changing " beep", "beep", "beep" ... into another notation, namely "*", "**", "***", ... and this "numbering off" of the particles automatically keeps track for the "star" of how many particles there are in the strip.

What results, of course, is the equivalent of the natural numbers. But such numbers are foreign to their way of thinking. Paradoxes arise because of breaking up, so to speak, the continuous messages - And, of course, another problem arises when they measure the part of the line each occupies, since they are now seemingly breaking up continuous space. All the logic and reasoning of which these particles are capable are brought to bear on explaining the nature of the "natural" numbers.

"How can one count parts of the line in such a man-

ner as this: * ** *** **** ...?" protests one of the critics. "You are missing out all the parts of the line between * and **! Actually it's really only * strip, and if you break it up into ever smaller and smaller parts, which you must do if you are going to count them all, then you'll never reach * strip. Of course, this must be correct reasoning since motion is impossible."

"But that's not what I'm doing at all," states the Star. "I'm not counting <u>all</u> the parts of the strip, I'm counting how many conscious particles there are in the strip. The problem you state is an entirely different one and would have to be solved by another system and another method."

Once the particles grasp the idea of the "game" and see how it can be applied in this problem, they see more applications for it. One application, of course, is to find what part of the line each occupies. This, though, is another game but they can use the same notation. Thus, since they know how many conscious particles in all there are, e.g. ******************************... ******, then each one can be said to be * of these parts, or */*****... ****************** of the line.

If the messages had merely remained a game with them, and they had only used them in playing this game, and not to find out how many particles there were in the

strip, then these messages would only have meaning as far as the game was concerned. However, once they find an application for these "numbers" then and only then do the numbers acquire a meaning for them over and above the game. Now they can explain "****", for example, as the Message that comes after the next one immediately after the next one immediately after the beginning message. This message can also be explained in the system of messages themselves. We can identify it in the series: * ** *** <u>****</u> ***** ....

Incidentally this far-fetched example also serves to illustrate another point, namely, that even though their "reality" is not really as they interpret it - For instance, that which seems fallacious to these particles, <u>viz</u>., their own discreteness, is what seems to be true to us, and what seems to be true to them, namely the flow of time, is what appears fallacious to us since we see it as motion (One particle being moved by another particle), that once their arithmetic is worked out and other sciences built upon it as a foundation, they can apply their arithmetic and scientific laws to their strip and find that they come out with satisfactory results. This seems to indicate that space and time, mathematical entities and scientific ones are not entirely mind-dependent but are related in some way with reality, even if this relationship is only one that gives meaning

to the words and does not serve as evidence for them.

The mathematics that has evolved is, of course, conventional and is based on the rules by which they have agreed to abide. In this world, too, it would look as if the irrational numbers would seem to be the natural ones and the "natural" and "rational" numbers as being the ones which have to be explained, and logical definitions given for numbers which cannot be demonstrated in physical experiments.

Finally, (the last of all these improbable worlds), suppose that a tribe of natives lived in a magician's world.

In this world things would keep appearing and disappearing. Now there would be no need to count objects as we do now. Such statements as "I saw a few cows in the field a while ago, but now there are none" would be sufficient. Mathematical statements as we know them (e.g. $1 + 1 = 2$) would be quite useless since the 1 might become 0 or 101 at any moment. Moreover, no two counts would ever be apt to give the same result since changes would be apt to occur in the very process of counting.

Here we would likely find that observation (assuming the observer remained long enough to make his observations) takes priority over a tallying up or a counting. There would be no use for ordinal numbers since nothing would remain stationary long enough for a "numbering off". As well, the same goat, for instance, might turn up in

the same column in several places during the count. In such a world, number-words as "none", "few", "some" and "many", would be all that would be required.

However, these bizarre worlds were not the imagined worlds of our imagined natives. Theirs was one where objects do have some stability and permanence. Their senses were such that they could distinguish one object or member of a group from another, and they could all recognize the same animal as the same one, and a different one as different from the other. The number-system that was invented served equally well the practical man and the philosopher.

However, even different living conditions alone might have produced a different number-system. For example, some other tribe living in a bountiful land where there is no need to worry about lack of food or the presence of enemies, and where trading and taxation are non-existent, or at least, unimportant, might have picked out some other common features of these same objects. Thus, if they had been aesthetically inclined, perhaps they might have been concerned only with lines and contours of the elements in the groups. Perhaps only with the colours, or with the sounds or smells. But these natives we are imagining, must eke out their existence as best they may. They are continually battling nature and each other, and they notice whether they

catch one fish or three fish. They keep track of how many cows they must trade for something they want from another native. They want to know how many men are opposing them in battle. This was a problem they had to solve, and since they had the number-sense the solution lay in developing the number language. They have merely developed the original concept of *ug* and *velly*, and they have made up words beyond these so that they can keep track of more objects in a collection.

It is for these reasons that this tribe has accepted the pebble-system, and later the number-words used in counting, as the criterion to determine how many objects there are in a group. Perhaps if such accurate information had not been so important to them they might have made observation the criterion of judging how many things there are in a group rather than the pebble-model and later the number-words. If this had been the case, then when discrepancies arose in a " count" all the counters would have to observe the group again and the majority's observations would be accepted. The counter who was "out" would be the one considered to be in error. But precise information was required for the reasons given above, and for these reasons the genius of the tribe invented "number" language. At first, at any rate, these people were not seeking the ultimate TRUTH - They were merely trying to find a simple system to serve them in their everyday needs in answering the question "How many?"

II. CONSISTENCY

At first, perhaps, the words the natives used were arbitrary. Some might still refer to "ug and ug and ug roca". Some might use "um-lah" when they had only ug and ug pebbles. However, once the system as set out by the inventor is publicly accepted by all members of the tribe, and likely by other tribes which come in contact with them as well, and it is demonstrated to all these people that one can count the sounds themselves, and that the umlath sound will always be um-lah, the decath sound, deca, and so on, or in technical language that the last ordinal number is the cardinal number of the group, then the order of these sounds will likely be accepted since it is seen that these sounds are merely an easy short-hand method of counting the members of any collection whatsoever - At least as far as they are concerned about collections at this stage. Once this serial order of the number-words is accepted then all the arbitrariness will be removed from the number words. This is much along the line that we accept standard measures today.

The objectors to the system are shown that this number language has a model on which it is patterned. This model is, of course, the pebbles. If the language is used like the model then no inconsistencies can arise. If the language is not used like the model then trouble is in store for the "mis-user".

"But why should this model be accepted?" asks one of the objectors. "Must we accept a system based on a model of the most worthless of all the things about us, namely pebbles?" Just because we can match other things with stones is surely no reason for us having to take this as a model of a number-system that will stand for all time!"

"Then what is your alternative?" ask the originators of the pebble-model and number-word system.

"One based on law and the rule of the gods," replies the objector. For example, here is a stick sent by god. You can see that it has many notches on it. Some of the notches are close together, and the cows or goats which match with them must be counted only as ug thing."

A second objector arrives on the scene. He not only disagrees with the originator, but with the First Objector as well. He says that the originator is trying to enforce his way of doing things - But it's a "free" country, and he will not accept such domination. He is free to say the words any way he wishes. After all, oo-pah things merely means that there are as many things in the group as a bird has wings, see-qua things means as many things as a man has fingers on one hand, and so on. There's no reason to put these words in any order, nor to follow any order when saying them. Anyone can say

the words in any order he wishes.

A third dissenter joins the group. He accepts the originator's pebble-model system, but he disagrees as to what is to count as <u>ug</u>. He says that, for instance, when you are counting cows, you must put <u>ug</u> pebble down for each <u>stage</u> of the cow you see. Thus, that cow is not the same cow it was when the sun rose this morning, nor will it be the same cow when the sun sets in the evening. You must count all the "stages" of each cow and goat and man, etc.

O#4, who has unobtrusively sat down with the others now speaks. "It is wrong to count the cows, the goats, men, and other things separately. If we do this, then others will desire to steal our things from us. We must consider all the cows in our land as <u>ug</u> cow, all the goats as <u>ug</u> goat, etc."

When these individuals count, all of them will arrive at different answers - And each one will have a reason for his answer. The originator uses his pebble-matching model and his ordered number-words. The first objector has used his god-given tally-stick. Objector #2 uses any order he wishes when he says the number-words. The next one (O#3) counts cow-stages, goat-stages, and all the other stages. O#4 disagrees with all the rest as to what is to count as <u>ug</u>.

An inconsistency has arisen in the number language.

All these natives are using the same language, and yet each comes up with a different answer - Each has a reason for his answer. There might be many other ways of counting the same group, and hence it looks as if any conclusion at all might follow when using this language.

However, the originator shows the usefulness of his system over the others'. He asks the others what they want to use such a system for. Is it important to each of them to know how many cows he has in his herd? They all agree that it is. O#1's method won't help them here indefinitely, for one or more of the notches on the stick may wear away, or merge with another, and whether or not this is the god's will it still won't keep track of cows if it does this while a count is being taken.

As for O#2, he is asked if he at any time has ever kept track of how many things there are by turning his fingers down one after another - He may even have a certain order for keeping this tally, by starting with the little finger of the left hand and continuing through to the thumb, and then starting with the thumb of the right hand and continuing through to the little finger. O#2 admits that he has done this.

"This is all I'm really doing with my number words," argues the originator of them. "I'm just saying these words in order. I say "ug" when you turn down ug finger (the little finger on your left hand), and I say "oo-pah" when you turn down the next one to your little

finger," and he continues thus through the series to deca.

O#2, being an intelligent man, acknowledges the other man's reasoning, and also acknowledges the usefulness in being able to say how many things there are when such a system as the number-words are used, instead of having to show how many things there are. O#3's system won't keep track of how many cows he can milk each night, whereas the originator's system is useful for this purpose. The originator's system is easy to standardize - That is, it can be simply explained as demonstrated above, and once these terms are accepted and rules made for the use of the system, then it is as precise an instrument as is needed by the natives in their calculations.

The natives can even check the system against itself in the following way.

They have all their soldiers line up in their regular positions. Each soldier, as well as the onlookers, knows how many men are between him and the starting point. The soldiers now number off. The ugth man in the line says "ug", the oo-path man in the line says "oo-pah". The me-tath man says "me-tah", and so on down the line - The deca-deca-deca-seentath man saying, of course, "deca-deca-deca-seenta". This numbering-off shows the natives that the last number said always tells them how many things there are in the group. The natives are assured

now that they can use these words to count how many animals, etc. that they have since the decath word is deca, then the decath cow must be deca, and hence they must have deca cows. It is to be noted that the natives get the meaning of the words from applying them in these ways.

As for the consistency and inconsistency of the system itself. It has been seen that the users of the language themselves decided upon what was to be counted as being consistent, and what not consistent. They agreed which system with its rules that they were going to abide by, and any other system that doesn't jibe with this one, as the objectors' systems above, from thenceforth will be considered to be inconsistent.

Now rather than worrying about the consistency of their system all that need concern them is whether or not they can use their system to express everything that needs to be said in it. So far, of course, they have been able to do this.

It was the usefulness of the originator's system, and the many uses to which it can be adapted which won both O#3 and #4 over to this system.

These five natives now accept the number-words, and decide to band together to keep their system of calculation, since they can see how they will be able to calculate much more quickly and rapidly with it than the

others who do not have such a system.

It is to be noted that such a system paves the way not only for abstract numbers, but also for the concept of successor of a number, and it's to this concept that we shall now turn.

III. THE CONCEPT OF SUCCESSOR

The concept of successor arose from the standard order in which they tallied items off with their fingers. This finger order is as follows: The little finger of the left hand comes first, the next finger to this little finger comes next, the next ug to the next ug to the little finger comes next, and so on up to the little finger on the right hand.

But this leads the natives to ordinal numbers. It is really the principle of ug, the successor of ug, the successor of the successor of ug, etc.

The genius of the tribe saw through this principle, and hit upon the plan of using different words for ug, the next ug after ug, the next ug after the next ug after ug, the next ug after the next ug after the next ug after ug, etc. His distinct words were merely abbreviations of this series.

The pebbles kept track of the ugs but that was all they did. They did not automatically tell their users how many ugs they had gone on from ug, the starting-point. The pebbles merely matched one for one with the members of a collection. The number words, on the

other hand do keep track of things. By knowing where he started to count (and this would be _ug_ until he understood the subtractive principle) and where he finished counting a native could discover how many _ugs_ had intervened.

These number-words are words in a regular series, much the same as the ticks of a clock are also in a regular series. There is a temporal relation between the ticks of a clock, and there would seem to be the same sort of relation between the number-words. Thus, 0 before 1 before 2 before 3 before 4....

In addition to this temporal relationship, the fact that the words occur in a regular series is important. 1 always comes after 0 (in the natural numbers), 2 always comes after 1, etc. If the number-words are not kept in this series, then not only is the temporal relationship lost, but the words lose their usefulness.

The usefulness referred to, is that each number-word implies all the number-words that come before it in the series. For example, the rule is that _two_ can be substituted for _one and one more_. _Three_ can be substituted for _two and one more_, and so we could continue indefinitely with the series. It is for this reason that the number-words automatically keep track of all that has been counted up to and including the one in hand.

Such a definite relationship and words are not unique

to the natural numbers. Any ordered series will show some sort of definite relationship between its terms. Huntington in his The Continuum (p.16) gives some examples of simply ordered classes.

> (10) The class of all instants of time, arranged in order of priority.
> (11) The class of all one's distinct sensations, of any particular kind, as of pleasure, pain, color, warmth, sound, etc., arranged in order of intensity.
> (12) The class of all events in any causal chain, arranged in order of cause and effect.
> (13) The class of all moral or commercial values, arranged in order of superiority.
> (14) The class of all measurable magnitudes of any particular kind, as lengths, weights, volumes, etc. arranged in order of size.[1]

Of course, one could arrange the natural numbers in an ordered series if one defined "less than" and then arrange the numbers in the order: 0 less than 1, 1 less than 2, etc. However, we'll use the temporal relationship since it is so immediate - Everyone alive has a pulse beat and so should be able to understand the meaning of this relation.

Since we are using this relationship, let us compare the ticks of a clock that keep track of time with the chugs of a motor-car which do not. In both cases there are audible sounds, and there is a before and after relationship. However, the chugs are not in a regular series. That is, there might be "chug, chug, chuuuuuuugggg, (There we lost track of some of them!) chug...."

The number-words, like the ticks of the clock, keep

[1] Huntington, E.V. The Continuum... 2d ed. Cambridge, Harvard Univ., 1955. p.16.

track for us, but a mumble-jumble of sounds do not, nor do the number-words spoken out of order. Of course, once we have the rules for our number-words, then we can apply to them to see whether or not we are following them, and then the temporal relationship is not essential to the serially ordered words. However, some such relation as this is necessary to get such an order in the first place, and later to give it meaning.

To return to our story and our natives, however. The number-words are gradually taking the place of the pebbles. Once the natives see that they can use the number-words like a measuring instrument that itself can be checked in the way described above, then the natives see that the number-words will automatically tell them how many things there are in a group or collection, and for this reason they are a great advance over the pebbles themselves which had to be displayed each time in order to show how many things they represented.

IV. NOTATIONS, OPERATIONS AND WHERE THEY LED

Some dreamers in the tribe might have thought that one could go on forever, ever saying ug after the last number-word spoken. It is unlikely at this time, however, that many were interested in such matters, since the groups or collections these people dealt with were small, and there was no need of even using words for numbers beyond fifty.

Over the years, this tribe gradually uses the pebbles less and less, substituting the number-words as the model for keeping track of how many things there are in a collection.

Moreover, as the years pass, our tribe increases in wealth, position and power among the other tribes. It becomes necessary now for an army to be built up in order to maintain this position, and for some of the scribes to be assigned the task of keeping track of the assets of the tribe.

The natives become more canny also in their transactions with others so that they may reap greater wealth for themselves and for their tribe. Some method of permanently keeping track of things is needed by the scribes over and above the number-words themselves.

At first, the scribes revert to pebbles, and tally marks. Sticks are gathered for the purpose, and each stick represents a certain trading transaction - The notches indicate how many cattle were sold for so many wea-

pons, and so on. These sticks are kept in the wise-men's offices, and are much respected by all individuals of the tribe.

To aid the scribes with these complicated matters, one of the natives got the idea of inventing money. This means of barter was arrived at by combining the bits of precious metal they had used, with the number words. Now the pieces of metal have distinguishing marks on them to show <u>ug</u>, <u>oo-pah</u>, <u>me-tah</u>, ... <u>deca</u> ... <u>deca</u>-<u>deca</u>-<u>deca</u>-<u>deca</u>...

However, the written numerals have not yet been invented, since the coin itself, its size and shape, are all taken into consideration, as well as the distinguishing marks for the number-words on the coins. It is to be remembered that the tally-sticks are still being used in the computations.

After many centuries, another genius arises in the tribe. He argues that since pens, ink and paper have all been invented, why not record tallies with these articles rather than on bits of archaic wood!

He goes on to say: "We can record, "/" for <u>ug</u> tally mark, or when we say "ug"; "//" for <u>oo-pah</u> tally marks, or when we say "oo-pah"; "///"; "////"; "/////" (or perhaps a picture of a hand); "//////; ///////; ////////; /////////; ////////// (or a picture of two hands).

Once the genius has this system at hand he notices that he can always go on from where he last left off just

by adding another "/". He sees this as an advantage to his system since trade is expanding so rapidly there is need for just such a system as this. I.e., a system that will always allow for adding ug more, to the last one written or spoken.

The system can be standardized so that others won't use it in other ways so as to get different results with the same notations. It can be easily taught to other people, and so there is no danger of it remaining a private system and hence of no value in commerce.

As for operations within the system. At first, the strokes can be used in the same way that the pebbles were used for operations of addition, subtraction, multiplication and division - That is, a mere combination or separation of pebbles in unequal or equal groups will allow for such operations.

The system of written numerals invented by this native served its users well as long as they confined their operations to what we call "adding" and "subtracting".

```
E.g. @@@#/
       @@ ////
   ------------
   @@@@@## which becomes @@@@@@.
```

The following example will illustrate subtraction:

```
     @@@@///          becomes @@@#////////
      @@@////                 @@@     ////
   -----------             ------------------
                                 #////
```

But when the natives attempted the short-hand method of performing these operations, namely when they attempted

to multiply and divide, they find these new symbols extremely cumbersome. Only a few able men ever really master the complexities of such a system. It's no wonder that these men are awe-inspiring to the average citizen.

A great shortcoming of this system was its lack of a symbol for naught. The founders had never thought of making a symbol, or even using a word, to represent or to say "no cows", for example. Yet such a word is a number word too, for by means of it one can answer the question, "How many?"

Because this new notation lacked a symbol for what we call "zero", and also lacked positional value (i.e., the value of the numeral depending upon its place in a column - the ones', tens', hundreds', thousands', etc.), it was exceedingly difficult to use it for complex computations (those involving large numbers). As a result of this, tally sticks, counting boards, and beads on frames were used, even by the scribes, to do much of their reckoning.

After a few more generations had passed by, news came to the tribe that another tribe, dwelling far away, could use numerals as easily as they could use their counting frames. These people had developed a symbol to represent the column of beads on the counting frame which was not used in some particular computation. The symbol, "~~a~~", was, at first, only a symbol to use like the empty column of a counting board, but by means of it one could

use a few symbols over and over again, since "ø" could be used to keep the others in their proper positions.

Again it is to be noted that no new principle is involved here. It is the same principle that was used by the natives when they used to drop a large pebble into the sack to indicate that all their fingers on both hands had been used in a count, and there were none left over, and hence no small pebbles were added to the sack. The new idea involved here is merely to show that all the fingers have been used, and that there are no more single *ugs* left, so ø is put down. To the left of it is put "/" which has the same significance as the big pebble had previously, or *ug* bead in the *deca* column has now. In the new notation, /ø showed that no beads were used from the *ugs* column, and that there was *ug* *deca*.

Now that ø was admitted to the "game" it should have been possible to add, subtract, multiply and divide with it. For example, when we add ///// and ø the answer is surely /////, just as if we put this many "/////" small pebbles in the sack and added *no* more. Similarly with subtraction. If we have a group of pebbles or strokes, and separate *none* from them, we still have the same group of pebbles. Moreover, no matter how many times we add *no* pebbles or strokes we still wind up with *no* pebbles or strokes. If we have no pebbles or strokes, then no matter how many times we try to separate no pebbles into

equal groups we still have no pebbles! What about trying to separate //////////// into a number of equal groups with <u>no</u> strokes in each group? This, of course, is impossible. Someone might argue that if we put just ug/oo-path (that is, if we separate <u>ug</u> stroke into <u>oo-pah</u> equal parts and take <u>ug</u> of them) of a stroke in each group then we would have <u>oo-pah</u> times as many strokes as we had to start with, and we could separate <u>ug</u> of the strokes into <u>um-lah</u> equal parts and put <u>ug</u> of these parts in, then we would have <u>um-lah</u> times as many strokes, and since the number of groups or strokes is increasing the smaller the part of a stroke we put in each equal group, then when we put in each equal group none at all we should have an infinite number of groups. However, since it is impossible to carry out this procedure, then we must say that it is impossible to divide with $\not{/}$.

It is to be noted that in these natives' world the application of mathematics had been in vogue long before the formalization of mathematics in any way, shape or form took place. The natural numbers are "natural" for these natives, and they were in use long before directions and formal operations were drawn up as such. In contrast with the Mobius Strip's inhabitants, here it is the natural numbers which are natural, and the irrational numbers that will have to be explained by the mathematicians and philosophers, since the latter cannot be seen to apply to reality.

The thought might even have occurred to someone that only the limited number of pebbles available would prevent a person from adding up more pebble to the stock-pile indefinitely.

At first, of course, such large numbers were of no importance nor interest to the members of the tribe. It was only when they began dealing with large numbers that they worked out the written numerals - That is, they transformed the notation of "/ // /// //// ..." into the new notation: / // /// //// # #/ #// #/// #//// * */ *// *//// *//// *# ... @@, etc. Now the prospect of ever larger and larger numbers began to interest many. The most daring of this brotherhood in his imagination went on to the infinite.

He saw that he could always add another "/" to any numeral he had, and hence there was no end to these numerals - the symbols for the numbers - and hence no end to the numbers themselves. The question, "What if I were to imagine all these numbers in a vast collection?" was forthcoming. Such a collection of numbers would be one in which there was no last number. Here one would be dealing with numbers that could not be counted, or matched with pebbles, and now one must be entering a different realm. In our terminology, this man was leaving the finite to set foot in the infinite - logically and not physically, of course. Soon others got interested in this "game" as well. Many of these individuals made the mistake of trying to read beyond the pieces in the "game" and of

those which can legitimately be brought into the "game". These individuals used the same rules, and the same system, for dealing with the infinite as they did with the finite.

Soon paradoxes were discovered (cf. Russell's paradoxes), which should have sounded a warning note to these "wise" men that something was not well with the way they were handling the infinite. Yet the majority of them continued to guess and read beyond what was already in the "game" or which could be brought to it legitimately. They were assuming that they could express in the system a number representing all the numbers of the system taken together. That is, they were assuming they could complete an infinite process, and then take all these numbers as one group and call this group or class of numbers a new number. They were attempting to express the unexpressible within the system, in other words.

These paradoxes showed, among other things, that the infinite cannot be handled in the same way as the finite. Different rules hold for infinite numbers. It is not the purpose, nor within the scope of this paper, to analyse the infinite numbers, so it will suffice here to give examples of such variance in rules.

Suppose that all the ordinary numbers were to change their behaviour. Instead of having $0^0 = 0$, we would find that $0^0 = 1$; $1^1 = 2$; $2^2 = 3$; $3^3 = 4$, etc. Now our common

everyday operations like addition, subtraction, multiplication and division could be done to one's heart content - but nothing would happen. One would be like a chipmunk on a wheel - Working away all day and getting nowhere! $3 + 1 = 3$; $3 \div 1 = 3$, $2 \times 3 = 3$; $3 - 1 = 3$; and even $5{,}263{,}456 \times 3 = 3$. It is said that one can get used to anything, so perhaps one would get used to such odd numerical behaviour in time - And if one did, then he would be right at home among the transfinite numbers, for that's just the way they work.

But now someone might ask just what these numbers stand for. For example, we can find an application for three, and five, and even seventy-five billions, six hundred fifty-three millions, four hundred seventy-eight thousands, nine hundred sixty-three - But what about Aleph-null, the cardinal number of the totality of natural numbers? Aleph-null can't be applied to anything in the universe as James R. Newman points out in his *Mathematics and the Imagination*.

The transfinite numbers stand for and only for numbers - They don't stand for anything else. Transfinite numbers can only play a role within the system - They can have no matter-of-fact role. The transfinite numbers, then, have no meaning apart from the system itself, and the system itself is a convention conceived by man. The transfinite numbers can only be expressed within this system.

As regards the strokes themselves. Each one can either in reality or in the imagination be put in a one-one correspondence with some real object in a concrete group or collection. But does such a correspondence always exist? Can this one-one correspondence always be demonstrated to exist? The answer to this question must certainly be "No". It cannot be demonstrated to exist in every case, although it can be said to exist. There is a difference, however, between it being logically possible to be able to say something exists and being physically able to demonstrate its existence.

A native genius, realized that if he made his system depend on such a correspondence as this existing, his system must then be justified by appealing to the actual world. In order to free his system from such bonds, the genius worked out a formal system which depends solely on symbols and definitions within the system, and directions outside it indicating what results can be inferred from a chain of reasoning within the system, and what symbols, or combinations of symbols, can be substituted one for another within the system.

Again one should note that the system cannot be compared to a snapshot of reality now. Rather it is more like a standardized measuring instrument which can be used to record thought processes (whatever they are - and this is not the place to delve into their nature). The

analogy then with a thermometer which measures a person's temperature, is a truer one for the number-system than is a snapshot of reality - The latter must be considered false.

The genius argues that once directions are given as to the sort of combinations of symbols which are allowed, and what substitutions can be made of symbols for symbols, and definitions are drawn up within the system, then the system is independent of checks with reality, and hence independent of space and time.

In this system, he has such symbols as: Sum(x,y),t: Subt(x,y),t: Mult(x,y),t; and Div(x,y),t. He then defines Sum (x,y),t, and the other operations by means of what were later called "recursive definitions".[1]

This genius constructed such functions as the above, e.g. Sum(x,y), where the variables (x and y) could only be natural numbers, and when natural numbers were substituted for the argument variables, the value of the function once again was a natural number. Moreover, he did not habve to assume the existence of a natural number which had a certain property, nor did the assumption have to be made that all the natural numbers had this property. (cf. with mathematical induction).

To give an example (and this is all that will be done in this paper), he could define the sum of adding ~~0~~ to <u>m</u> thus: Sum(~~0~~,m) = m

---

[1]See Goodstein, R.L. <u>Recursive Number Theory</u>. Amsterdam, North-Holland pub., 1957.

Sum(n + /,m) = Sum(n,m) + /.

The second equation told him how to find the value of the addition of n + / to m when the value of the addition of n to m had already been found. He is thus enabled to find the values of the function for n = ∅, n = /, n = //, etc.

He can readily show how ∅ works in his system, thus: Sum(x,∅),x; Subt(x,∅),x; Mult(x,∅),∅; Div(∅,x),∅. He can explain this last function by means of the inverse operation, multiplication, since Mult(x,∅),∅. But Mult(m,∅) will always give ∅ and never x and hence he argues that Div(x,∅) is impossible. All that can be said in regard to Div(x,m) is that it's an increasing function if one substitutes parts of whole numbers (proper fractions) in place of m.

What is to be noted is that the inspiration for such a system was already present in the tribe's method of keeping track of things. The very use of pebbles themselves was the first step on the road to the invention of an instrument that could operate independently of checks with cows, goats, apples and other items in the world about them.

A critic now asks the inventor if he can express everything that needs to be expressed in this system. Can he, for example, express the greatest number of all?

The inventor of the system replies that he could certainly make a symbol for the greatest number of all, but it would be impossible to complete the infinite series and arrive at the greatest number of all, and hence such a number could not be expressed within his system.

Before proceeding with new material it might be as well to sum up what has already been learned by the natives.

We have seen how the natives first made use of pebbles in their counting and computations. A refinement occurred in this method when they allowed the large pebbles to represent <u>deca</u> objects, and the small ones to represent the <u>ugs</u>. By means of this system they were enabled to arrive at the cardinal number of any group, E.g. "@@@///" cows.

Not only were the cardinals invented, but the ordinals paralleled the development of the cardinal numbers. This was due to the fact that man used his fingers for tallying up how many objects there were in any group. Since a certain order was preserved in this finger-counting method, and since the ordinals are merely the numerals arranged in order, we see that the ordinals were used to enable man to arrive at the cardinal number of a group.

The pebbles and the pebble-system enabled these natives to tell when two or more groups had the same <u>number</u> of members, and hence gave them the <u>equality</u> concept. If the groups differed in number they could tell which was the greater group and so they got the concept of "greater than" or "more" and "less". By means of the pebbles alone they found out that when two or more groups are combined that counts of each group separately gave the same result as a through count of the large group composed of the two or more smaller groups. Moreover, they dis-

covered that the order of counting or combining was immaterial. Thus, it was immaterial whether they combined Group A with Group B or *vice versa*. Nor did it matter if they combined Group A and B first and then combined this larger group with Group C, or if they combined Groups B and C first and then added Group A.

Such experiences led to *laws* being made as follows: "*The Associative Law* which means that the terms of an expression connected by plus or minus signs can be grouped together in any way.

Commutation which means that the terms or parts of an expression which are connected by plus or minus signs can be written in any order....

Distributive Law means that addition and subtraction can be performed in any order."[1]

Later these laws were proved in the formal system.

We have also seen that computations could be carried out without number-words, and that with the advent of written symbols (including Ø) that the numerals designating the natural numbers could have been worked out.

There is no need for axioms to be presented nor the deductive system used either in any of this evolution. The laws made for the system were empirical in that they appealed to experience both to obtain them and to apply them. The system was not set up as a logical system with a set of axioms and formation and transformation rules.

---

[1] McDowell, C.H. *A Short Dictionary of Mathematics*. *op.cit.*

Rather, the laws were invented to jibe with what had been learned through experience, and they were followed unconsciously before being formulated in words and rules. Even Peano's axioms are implicit in the pebble-system itself, and hence it is unnecessary to postulate them.

Once the concept of "ø" was grasped, then to the question "How many gold bricks have you?", for instance, a native could reply be showing an empty pebble-sack. Hence, *P1. ø is a number.*

When someone adds *ug* pebble to a group of pebbles, this allowed him again to answer the question "How many?". Their system was such that they could always add *ug* more pebble to enable them to answer the question, *How many.* This gives *P2* - *The successor of any number is a number*.

Since *ug* pebble is added to the sack for each new "number" then this very system implies *P3* - *No two numbers have the same successor.*

To the question, *How many*, they can show an empty sack, and that is that, as far as this system goes - They can show no fewer pebbles than none at all. Hence, *P4* - *ø is not the successor of any number*.

Finally it can be seen that one can use "ø" to answer, "How many?", and when one can use *n* pebbles to answer the question, "How many?", then one can also use *n* + 1 pebbles to answer the question, "How many?", and so every bunch of pebbles can answer this question. This,

of course, is Peano's P5.

Some of the natives balk at P5. One of them states that it reminds him of a game with thin blocks of wood. In this game all the blocks are placed on end within range of each other. Someone pushes the first block and they all eventually nudge one another and fall down. But with P5, one must imagine an endless procession of such blocks. Now this man says that he is satisfied that the blocks he can see are falling, and he is satisfied that the blocks he can go and check on if he wants to will fall - But what about the blocks that are way beyond the hills and across the lake - Will they fall too? For some of them, it is true that he can never know for sure. Only in the same way as he assumes that tomorrow will follow today can he assume that all the blocks will fall, and until he learns differently, build his conclusions on such assumptions. This is the only assurance that he can offer himself and others. Needless to say, the controversy over P5 still goes on.

There is a fallacy in this man's reasoning however. He is confusing the physical possibility of an innumerable number of blocks falling down with the logical possibility of saying that an infinite number of blocks will fall down. That is, he is confusing the logical possibility of always being able to say that one can knock one more block down in an infinite process and of physically

being able to do the same thing.

The question may arise as to the value of the number-words since apparently they are not essential to the working out of the numerals which give us the natural numbers.

The value of the number-words we have seen to lie in their practical usefulness. They appeared in the system as short-cuts that sped up man's calculations. The number-words not only cut down the amount of calculation, but they allowed the natives to get a better grasp of the whole situation, and so to understand the system better and to develop it. Words and explanations in words are of practical use to the system. As we have previously noted, it is the whole system that must be considered and not just the signification of the word itself. Words tie mathematics in with the rest of knowledge, and so make mathematics more than just a game with symbols. We can apply our mathematics to our world about us. If mathematics had just developed as a formal system, isolated from all other fields of knowledge, it is extremely unlikely that it would have reached its present stature.

Throughout this tale we have seen that it is the practical man who has developed mathematics, and given to the tribe the pebble-system and later the number-words. It was not at first, at any rate, the mystic seeking after ABSOLUTE TRUTH.

As succeeding generations of natives make use of

the number-system, and extend it and adapt it to their uses and needs, the numbers and the numerals may be used more and more on their own and less and less use is made of material objects with which to explain their meaning. The users of the language now understand the words, and they can show how they are used, if necessary, by means of examples. However, generally now it is not necessary to explain their meaning by means of words which they can understand directly from their own experiences - That is, what they see, what they feel, what they do. The number-words now have meaning for the members of the tribe, and they quite forget all the details which were intentionally omitted in order to arrive at such words in their language. Some individuals, however, remember these details and they understand the number-system. They realize that the others are not seeing the whole picture when they think of the numbers themselves as independent entities - Independent of the whole system.

SECOND SECTION THE NATURE OF NUMBERS

Now that we have dealt extensively with the origin of the natural numbers, let us turn to our number system itself and see if the question, "What is number?" is meaningful.

Suppose someone is sitting in his room repeating the natural numbers to himself. Let us assume that both he and we know that he is repeating the natural numbers. We shall not consider the conditions which gives us such knowledge, or even what it means that we should know this. This is all taken for granted.

Suppose as well that he's not counting things in his room. There is no collection of things before him. He is not imagining something which he could count. It might be said that he's learning what the numbers are. He is also trying to learn the order in which they ordinarily come.

There's a difference between asking, from his point of view, what the first number of the series is, and asking what the number _one_ is. It is possible to answer the first question by saying "The first number of the series is one". But how would one answer the second question, _viz_.what the number one is?

Suppose we say to the boy: What you want is not just to know how to go on from wherever you are or how to count up higher than you've yet been able to do, but

you also want an idea of what makes possible the sort of thing you're doing. You have been using certain numerals in order to repeat certain numbers. These numerals, like words, one can hear you pronounce and watch you write. One can also use them in one's head without saying or writing anything.

Suppose he asks whether a numeral one uses is the name of something. Just as, in studying geography, he pronounces the words "Brazil", "Argentina", "Mexico", "Canada" and in pronouncing those words gives the names of various countries, he might pronounce the numerals "one", "two" and "three" and wonder in pronouncing those numerals whether there are things whose names he is also giving.

In the one case the student is merely pronouncing words, but here the words are names. Each one is the name of a particular country. If we stop the student and ask him about one of them, say "Brazil" he might reply: "Oh yes, I am repeating names of countries, and although I'm merely learning the names of twenty countries, I am able to tell you that Brazil is a country in east and central South America. It has an area of 3,286,169 square miles, and its capital is Rio de Janeiro."

Not only can we check the veracity of the student's statements in a geography text or atlas, but we can even visit Brazil to see for ourselves. Brazil, designated

by the name "Brazil" exists in space and time. What about numbers - How can they be said to exist?

Perhaps we can tie the numbers up with the numerals in some relationship which will give them a concrete foundation. <u>One</u> designates something concrete too - It designates a concrete symbol - Here it is - <u>1</u>. Numbers may be like the names of countries after all. Indeed, we can write out all the symbols representing the numbers, and point them out in the same way we can point any other concrete objects out. There is nothing abstract about these symbols. We can feel them, move them about, and generally use them as we do any other material object.

But surely there is a difference here. Let us say a nuclear blast occurs and literally wipes the Island of Oahu off the face of the earth. Now there is nothing that " Oahu" designates when someone says this word. It only designates an island that existed in space during the time of ____ to ____. But the same would not hold true if the same nuclear blast destroyed every one of the numerical symbols. The numbers "one", "two", "three"... would still be meaningful to intelligent persons who understood the number language, no matter where these individuals might be. The numbers do not depend upon experience in space and time as does Oahu or Brazil. If they did then we'd always have to use the same numeral to express the same number. Thus "/~~1~~" would always have

to express the number <u>ten</u>, just as "Oahu" designates a certain island in the Pacific and no other. It would seem that rather than the number <u>one</u> designating the numeral <u>1</u> it's the other way round. <u>1</u> designates "one".

Possibly then, the Platonists are right in arguing that numbers exist, and that all man can do is to discover these abstract entities!

To return to the lad and his lessons, however - What answer can he give to us, or explanation can we give to him when the problem is raised as to the nature of the numbers which he is repeating? They certainly can be distinguished from numerals as is shown above.

The lad might reply that he is merely repeating a sequence which he has been taught, and that each word he says implies its predecessor and 1 more. Moreover, he doesn't need material objects about him to which to relate these numbers in order to get this series.

He might even argue that he could have caught on to such a system if he had been suspended sightless in space and hence unable to distinguish one star or planet from another. The very throbbing of his pulse could have been the inspiration for such a system. Pulse beats are regular, and they occur in a regular order so that the one which is just pulsing now is the immediate successor of the one that just beat previously, and the immediate predecessor of the one that is just going to beat.

These pulse beats are arranged in an order of priority. There is a temporal relationship of before and after, and the concept of betweenness. The boy might argue, or someone argue for him, that any such ordered series would give him the natural numbers. Hence there's no need to delve into anthropological findings to discover how natives kept track of their cows, goats or sheep. In other words, a paper such as this one is of no value whatsoever in explaining numbers and numerals!

It is true that pulsebeats, like instants of time, are in a regular series, but this isn't enough in itself to give the boy, or anyone else, the natural numbers. However, such an argument as this leads to the notion that mathematics is *a priori* in that time as compared to space, seems to us to be internal somehow, and hence universal. Thus the boy might argue that his very life depends upon moments of life (and the pulse beats emphasize this), and the very fact that one moment follows another in a regular succession is enough to give him the natural numbers.

But is this really enough? How does he know that there are any other beings, and if there are that they are constructed exactly as he? What about the conscious bits on the twisted strip, for instance? Even if this is a far-fetched example, still it can be argued that since this boy is arguing from intuition, that intuition has

not proven itself trustworthy in the past. For example, consider the intuitive notion of Euclidean space.

As for the worthlessness of the long account of the development of mathematics - It seems to the writer that this account, based as it is to some extent on historical facts, shows the length of time it took man to develop mathematics. Surely if it were an intuitive matter common to every living person such as the falling apart of a life moment, mathematics would have evolved relatively early in man's history.

The fallacy in this lad's reasoning seems to be the confusion of any ordered series with the ordered series of the natural numbers - But what is needed in addition to sensing the pulsebeats, or in being aware of the falling apart of a life-moment, is some sort of constructions that can somehow be recorded. These constructions, or transformations of them, are needed by man in order that he can have a suitable symbolism for the developing of mathematics. The pulse beats, the life moments, and any other collection one cares to name may be there - Even all spread out for man to see - But without the rules that change the members of the collection into a notation - a symbolism - the writer fails to see how mathematics could ever have developed. What is needed to develop even the natural numbers, in her opinion, is noticing that there is a beginning construction, and then

successors - The constructions for these successors must be able to be identified one from another.

It would appear, then, that rather than having the natural numbers resting upon an intuitive basis, that some such experiences as have been described in this paper are necessary, and constructions made. Moreover, since it is these very symbols which designate the numbers that are used in mathematics[1], mathematics must be <u>synthetic</u> <u>a</u> <u>posteriori</u> to this extent.

However, once we have this ordered series, and the notion of <u>equality</u>, <u>greater</u> <u>than</u> and <u>less</u> <u>than</u>, then we can go ahead and construct a mathematics which is independent of experience.

To illustrate this point, consider the following lesson in counting.

The teacher uses the following method for her lesson. First of all she writes out the following symbols on pieces of cardboard - One symbol on each piece of cardboard: ---- / - --- */ *--- /--- /- /-- -- * and so on. The teacher then gives the child a certain number of objects such as apples. The child is required to pair off apples with pieces of cardboard in any order he wishes - One piece of cardboard for one apple. Now when asked to display by means of the numerals how many he has, the child must show the teacher all the numerals - Rather, all the pieces of cardboard he has used in this matching

---

[1] The writer is referring only to arithmetic here.

process.

The next step in the lesson is to arrange the numerals in their order of succession, thus: - -- --- ---- / /- /-- /--- /---- * *- *-- *--- *---- */ */- */-- */--- */---- **, etc.

It doesn't matter what kind of marks are made for these numerals of course, although now it is important that one can see how to always go on from the last one.

This time the child must always put down first the piece of cardboard with "-" on it to keep track of the first apple he counts, "--", must be put down to keep track of the first apple and the <u>next</u> one, Then comes "---", "----", and so on until he has accounted for all the apples in the group.

The child repeats this procedure a sufficient number of times to see that he always ends up with the piece of cardboard with the same marks on it, E.g. */---. The teacher as well as other pupils also repeat the procedure with the same group of apples. The child thereby discovers that no matter how many times the process is repeated nor by whom it is repeated (as long as the individual knows the rules) the same <u>numeral</u> always turns up for the count of this group of apples. Of course, we must also add the <u>proviso</u> that the group remains stable - That is, no apples rot, and everyone who counts knows what is to count as one apple, and what as two.

Now when the child is asked to show how many apples he has, he can point to this numeral: */---. This numeral is unique - It is the one following */--, and the one immediately before */----, and it can be none other. Of course, if someone else just walked into the room and asked the child how many apples he has and the child pointed to this numeral: */---, this would mean nothing to the inquirer - It is not intuitively evident what is meant by this, or by any other system - Certainly not by a series of strokes: ///////////////////////////////////, for the identifying reasons stressed in the early part of this paper.

The intruder may say that we are confusing an exact concept (the marks which we conceive to be exact) with empirical concepts - the apples, trees, etc. He may argue that these things in nature are not always exact. For example, how could one measure the potatoes in a heap of potatoes put on his dinner plate, or even how many bald men there are in the auditorium, when the baldness of some of them is just at a crucial stage.

"But this is not the purpose of the system," the teacher explains. "The system is not one that can be compared with a snapshot of reality. Rather, we want to know how many things there are before us in that group." The teacher goes on to explain that when we want to know badly enough how many things there are, then we use our measuring instrument - And this is the instrument we are

using now: - -- --- ---- / /- /-- /--- /---- * *- *-- ...
Since we can distinguish certain objects one from another as is pointed out in the anthropological section of this paper, and we can define beforehand what is to count as one object and what as another distinct object (cf. the controversial bald-headed man), then we can apply our instrument to measuring this and not to measuring that.

"In the same way," the teacher goes on to explain, "now that I've had such experiences I am inspired to make a formal system: $\phi(m,n),r$. When you ask me what is the result of adding $\underline{m}$ and $\underline{n}$, for example, I can say "$\underline{r}$". If you don't understand the system, of course, then you don't know what I mean by this. However, I can define such familiar operations as: addition, multiplication, subtraction and division in this system. Once I show you the rules, operations and symbols of the system, and show you that $\underline{r}$ is the result of Sum(m,n) then I have shown you how you can express $\underline{r}$. Moreover, when I substitute members of the ordered series: 0 - -- --- ---- ... for $\underline{m}$ and $\underline{n}$ the result will be another unique member of this series. I can even go on to develop most of the mathematics in use today and with no fear of the dread antinomies arising that so plague the logicists."

"Then what's the point of having number-words at all?" inquires the puzzled inquirer.

"This will be illuminated in the next step in the lesson," continues the teacher.

The teacher then proceeds to teach the pupil to say "one" when he speaks of a single apple, or when he refers to the numeral "-", or by means of transformation rules, the numeral 1. "Two" is to be said when he speaks of one apple and one more, or when he refers to the numeral "--", which becomes 2 with the transformation rules which changes "--" to "2", etc.

The child is now being taught the number language. He is being taught the language to use when he wishes to talk about how many objects there are in a group or collection. He is also being taught the role that these number -words play in our language.

The child is being taught both the numerals and the numbers in this lesson in counting. When he first learns these number words and how to use them correctly in sentences, such as:"There are five apples in that bowl," he is learning matter-of-fact sentences. He can check with the objects to see if he is using the words in the way in which they have been taught. He can even check the words against themselves to see if they always come out according to the way they have been taught.

Later on, however, as in the case of the student counting, such empirical checks are not needed. Instead one can appeal to the rules: one after zero, two

after one, three after two, and so on.

Now when the student is asked what the number one is, he can give several answers. He can say it is the first number in the series: "one, two, three, four ...". He can also show where it can be used in the formal system, e.g. Sum(1,0),1, and he can show how it can be used to apply to the world about us - For example, "That little girl has eaten one apple from the bowl of apples." These are the various roles that the number-word <u>one</u> can play. We can express <u>one</u> in these various ways. So the student goes on to state that rather than asking him the question,"What is the number one?", the question should rather have been; "How can one express the number one?" He says that he is unable to answer such a question as: <u>What is number</u>, but the question: <u>How can number be expressed</u>, is both intelligible and meaningful.

The value of the number-words lies in the fact that by means of them one can relate other words and sentences in one's language. Such words as: <u>none</u>, <u>some</u>, <u>few</u>, <u>many</u>, can all be related with the number words. This allows us to assimilate our knowledge, and may inspire some to go on to further inquiries into new fields of research.

This section of the paper shows us the reason why the number-words seem to have a non-empirical intuitive nature. Thus, no matter what space is blown up, as long as a man survives with the sense to beat time, or

sense the passing of a life moment, and is able to discern the ordering of the series (the relation of <u>before</u> and <u>after</u> between each member of the series), and also if he hits upon the notion of using transformation rules to transform one ordered series to another that can be used for measuring <u>how many</u>, then he can get the natural numbers. He, of course, might not develop the same numerals to designate the numbers, but that would be a minor matter, since any suitable characters would enable him to play his game as long as he knows what the rules of the game are. If he has forgotten the rules, then some kind of strokes, or some such constructions, must be made so he can arrive at a symbolism suitable for counting purposes.

This imagined lesson in counting also shows how a child is taught the application of the numerals before being taught the number-words. It is the application which gives the meaning. If the child were taught only the mathematical formulae and nothing else, anything which we call the <u>sense</u> of the whole system would escape him. The child might understand how to read the formulae, but he wouldn't know how to interpret them in his world, and as well, tie them in with the rest of his knowledge. The number-words, then, serve to integrate arithmetic with the rest of knowledge.

This paper has shown that it was man's need to be able to show and to say precisely, and in a standardized way, just how many things there were in a group or collection that gave rise to number-words. What was present before the numbers, however, was man's primitive number-sense which he may even share with the higher brutes. As well there were collections of distinct objects.

We might very aptly compare the invention of numbers with the invention of thermometers.

What was there before the invention of thermometers was man's ability to distinguish different degrees of heat and cold, and of course there were substances with distinct temperatures.

At first man could only inconclusively show and say how hot or how cold anything was. "It's too cold to go hunting today." "That milk is too hot to drink", etc. To be more precise than this someone invented and standardized thermometers. We have seen that the natural numbers came into being in much the same way.

The thermometer does not give one a photographic picture of reality, nor do the numbers.

The temperature of the air, or the liquid, or the person, is there before anyone reads it on a thermometer. The thermometer is only a measuring instrument which contains a liquid which rises in the tube as does the temperature. Once the scale is marked on the instrument

and its applications are shown, then everyone who understands the language and sees how it is applied, understands the meaning of the words designated by the notation on the thermometer.

In much the same way, the number of the collection is already there before its objects or elements are counted. Man has merely made an instrument which measures the number of the collection in conformity with the number of elements that are added or subtracted from the collection. Of course, this <u>instrument</u> can be used to measure other collections providing they are applicable to its notation, in the same way that the thermometer can be used to measure other liquids or substances as long as they are applicable to the kind of thermometer it is. (E.g. A weather thermometer is unsuitable and can't be used to measure the temperature of boiling fudge).

Moreover, a thermometer-like instrument which could not be applied to reality, could be made solely for the purpose of playing a game. In this case, the directions for playing and the terms of the game itself have meaning only within the game itself. In the same way, a formal system can be made which is quite divorced from reality. The terms and operations are terms and operations within the game and nothing else.

However, just as the temperature readings of our present-day thermometers have applications in reality, so

the words designated by these readings have meaning apart from the instrument itself. Indeed it is these roles as well as the roles of the words within the system that give these words their meaning.

In the same way, it is the application of the number-system which gives meanings to the words designated by the numerals. The fact that number-words have matter-of-fact roles as well as roles to play within the formal system itself allows us to interpret its formulae in various ways, and in various fields outside the formal system itself.

In the same way, the weather thermometer cannot be used to measure the temperature of boiling fudge, so the natural numbers cannot be used to measure the number of "classes" to which they don't apply. We must be careful how we interpret mathematical formulae. If they are interpreted in a way that is inconsistent with the ordinary usage of terms in other fields, then, of course, inconsistencies will follow. To date the greatest source of trouble in interpretations seems to have been the little word "all".

Finally, this paper has attempted to show the conventional character of mathematics. It has been shown that even what is to count and what is not to count as consistent is a convention.

A SELECTED BIBLIOGRAPHY

Britton, K. "Counting ..." (In *Journal of Philosophy*. 44:309-18. 1947)

Conant, L.L. "Counting" (In Newman, J.R., ed. *The World of Mathematics*. N.Y., Simon & Schuster, 1956)

Dantzig, T. *Number, The Language of Science*. N.Y., Doubleday, 1954.

Eves, H. *An Introduction to the Foundations and Fundamental Concepts of Mathematics*. N.Y., Holt, Rinehart & Winston, 1961.

Goodstein, R.L. *Constructive Formalism*. Leicester, Univ. College, 1951.

Goodstein, R.L. *Recursive Number Theory*. Amsterdam, North-Holland pub. co.,1957.

Heyting, A. *Intuitionism, an Introduction*. Amsterdam, North-Holland pub.co.,1956.

Huntington, E.V. *The Continuum*. 2d ed. Cambridge, Harvard Univ., 1955.

Kline, Morris. *Mathematics and the Physical World*. N.Y., Doubleday, 1963.

Logsdon, M.L. *A Mathematician Explains*. Chic.,Univ. of Chic., 1960.

McDowell, C.H. *A Short Dictionary of Mathematics*. N.Y., Phil. Library, 1957.

Newman, J.R. *Mathematics and the Imagination*. N.Y., Simon & Schuster, 1940.

Newman, J.R.,ed. *The World of Mathematics*. N.Y., Simon & Schuster, 1956.

Quine, W.V.O. *Word and Object*. Cambridge, M.I.T.,c1960.

Schilpp, P.A.ed. *The Philosophy of Bertrand Russell*. Evanston, 1944.

Waismann, F. *Introduction to Mathematical Thinking*. N.Y., Harper,c1951.